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The Gibbs Phenomenon for Multiple Fourier Integrals

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Restrict a smooth function to a domain bounded by a smooth surface. We study the summability of the Fourier integral of this function at points near the boundary of the domain.

The Gibbs–Wilbraham phenomenon describes the behaviour of the partial sums of a Fourier series in a neighbourhood of a simple discontinuity of the function expanded. See, e.g., [6] for some history and references. It is well known that this phenomenon of "overshooting" arises not only in the theory of trigonometric Fourier series, but also in a variety of other expansions. In this paper we want to investigate the Gibbs phenomenon for the summability of multiple Fourier integrals.

Cramér proved that for the Cesàro $\mathbf{C}^{\alpha}$-summability of ordinary Fourier series there exists a critical index α_0, $0<\alpha_0<1$, with the property that for $\alpha<\alpha_0$ the $\mathbf{C}^{\alpha}$-means of the Fourier series of discontinuous functions of bounded variation show the Gibbs phenomenon, but for $\alpha \geqslant \alpha_0$ the overshooting disappears. See [4; 10, III.11].

For the Bochner–Riesz summability the situation is different: the Bochner–Riesz means of any order of the Fourier series of discontinuous functions of bounded variation present the Gibbs phenomenon. See [2]. This is a bit surprising since the Cesàro and the Bochner–Riesz summability methods are equivalent. Cheng also investigated the Gibbs phenomenon for the Bochner–Riesz means of double Fourier series of functions of the form $f(x, y)=f_1(x)\cdot f_2(y)$, with $f_1(x)$ and $f_2(y)$ of bounded variation over a period.

Golubov studied the Gibbs phenomenon in N-dimensions for the Bochner–Riesz means of order $\delta>(N-1)/2$ (the "critical index" for the Bochner–Riesz summability) of multiple Fourier series and integrals of functions of "Φ-bounded variation" with discontinuities on a sphere or on a hyperplane. See [5].

However, the main motivation of our work comes from a series of apparently unnoticed papers by Weyl published in 1910. See [9]. In these

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papers the author applied his experience with various eigenfunction expansions to several examples of the Gibbs phenomenon. In particular, he considered the following situation.

Suppose that on the two dimensional sphere $\{x^2+y^2+z^2=1\}$ we have a smooth closed simple curve γ that divides the sphere into two open sets A and B, and suppose that we have two smooth functions $f_A(P)$ and $f_B(P)$ defined on $A\cup\gamma$ and $B\cup\gamma$, respectively. Let

$$f(P)=\begin{cases} f_A(P) & \text{if } P\in A,\\ f_B(P) & \text{if } P\in B,\\ \frac{1}{2}(f_A(P)+f_B(P)) & \text{if } P\in\gamma. \end{cases}$$

Finally, let $\{S_nf(P)\}$ be the nth partial sum of the spherical harmonic expansion of the function $f(P)$. At any point of the curve γ we can associate a great circle normal to γ, and the nature of the convergence of the spherical harmonics $\{S_nf(P)\}$ on this great circle is the same as the convergence of the trigonometric Fourier series of the restriction to the great circle of the function $f(P)$. That is, for the two-dimensional spherical harmonic expansions of functions with discontinuities along a smooth curve, we have qualitatively and quantitatively the same Gibbs phenomenon as in one dimension. If the curve γ has a corner or a cusp, the behaviour of the spherical harmonic expansions is more complicated.

It is the result of Weyl that we want to extend from the spherical harmonic expansions on the two-dimensional sphere to multiple Fourier integrals in R^N. In Section 1 we shall consider the Gibbs phenomenon for general summability methods of multiple Fourier integrals, and in Section 2 we shall study in more detail the Bochner–Riesz means of double Fourier integrals. By the way, we shall see that a precise analogue of Weyl's result may fail in dimensions three or higher.

1

Let $S(x)$ be a "kernel" on $\mathbb{R}^N$, $\int_{\mathbb{R}^N} S(x)\,dx=1$, and let $\{S_R(x)\}=\{R^NS(Rx)\}$ be the associated family of approximate identities. We shall study operators of the form

$$\begin{aligned} S_R*f(x)&=\int_{\mathbb{R}^N} R^NS(Ry)f(x-y)\,dy\\ &=\int_{\mathbb{R}^N}\hat{S}(R^{-1}t)\hat{f}(t)\exp(2\pi ix\cdot t)\,dt. \end{aligned}$$

To introduce our first result we briefly recall why and when the Gibbs phenomenon arises in one dimension.

Suppose that $K(x)$ is a kernel on $\mathbb{R}$, $\int_{\mathbb{R}} K(x)\,dx = 1$, and that $K_R(x) = R\,K(Rx)$. Then, if $\chi_{(-\infty,0)}(x)$ is the characteristic function of the interval $(-\infty, 0)$, and if $x > 0$,

$$K_R * \chi_{(-\infty,0)}(x) = \int_{-\infty}^{0} R\,K(Rx - Ry)\,dy = \int_{Rx}^{+\infty} K(y)\,dy.$$

Suppose there exists an $A > 0$ such that $\int_A^{+\infty} K(y)\,dy < 0$, then in a neighbourhood of $x = 0$, the point of discontinuity of the function $\chi_{(-\infty,0)}(x)$, the graph of the means $\{K_R * \chi_{(-\infty,0)}(x)\}$ does not converge to the graph of $\chi_{(-\infty,0)}(x)$. If some convergence and localization principle holds, then the Gibbs phenomenon arises not only for the means $\{K_R * \chi_{(-\infty,0)}(x)\}$, but also for the means $\{K_R * f(x)\}$ of every function $f(x)$ with simple discontinuities.

In the following we shall often write $x \in \mathbb{R}^N$ as (x_1, x_2), with $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}^{N-1}$.

THEOREM 1. *Let $m(t)$ be a bounded even function on $\mathbb{R}$, with the property that $(m(t)-1)/t$ is a locally integrable function with Fourier transform vanishing at infinity, and suppose that the two kernels, one on $\mathbb{R}$ and the other on $\mathbb{R}^N$,*

$$K(x) = \int_{\mathbb{R}} m(t) \exp(2\pi i t x)\,dt,$$

$$S(\mathbf{x}) = \int_{\mathbb{R}^N} m(|\mathbf{t}|) \exp(2\pi i \mathbf{t} \cdot \mathbf{x})\,d\mathbf{t},$$

are in $\mathbf{L}^1(\mathbb{R})$ and in $\mathbf{L}^1(\mathbb{R}^N)$, respectively.

Let D be a domain in $\mathbb{R}^N$, bounded by a smooth simple closed surface ∂D. Assume that the point $P = (1, 0)$ lies on ∂D, and that the vector $\mathbf{n} = (1, 0)$ is outward normal to ∂D in P. Let $F(\mathbf{x})$ be a smooth function on $\mathbb{R}^N$, and let

$$f(\mathbf{x}) = \begin{cases} F(\mathbf{x}) & \text{on } D, \\ \tfrac{1}{2} F(\mathbf{x}) & \text{on } \partial D, \\ 0 & \text{on } \mathbb{R}^N - \bar{D}. \end{cases}$$

Then, if the point $\mathbf{x}$ is on the outward normal to ∂D in P and close to P, i.e., if $\mathbf{x} = (1 + \varepsilon, 0)$ with ε a positive small number, we have that

$$S_R * f(\mathbf{x}) = 2 f(P) \cdot \int_{R(|\mathbf{x}|-1)}^{+\infty} K(s)\,ds + E(\mathbf{x}, R),$$

and $E(\mathbf{x}, R)$ converges to zero as $R \to +\infty$ uniformly with respect to $\mathbf{x}$ in a suitably small neighborhood of P.

Note. $2f(P)$ is the "jump" of the function f at point P.

In some sense the two kernels $K(x)$ and $S(x)$, one on $\mathbb{R}$ and the other on $\mathbb{R}^N$, have the "same" Fourier transform, and this theorem means that under suitable assumptions the behaviour of the means $\{S_R * f(x)\}$ for x in a line ℓ is qualitatively and quantitatively the same as the behaviour of the means $\{K_R * H(x)\}$ where $H(x)$ denotes the restriction to the line ℓ of the function $f(x)$. The idea of the proof is straightforward.

We first prove the result when the domain is the unit ball centered at the origin $B(O; 1)$ and the function is the modified characteristic function of this ball,

$$\chi(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ \frac{1}{2} & \text{if } |x| = 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

Then we approximate the domain D around the point $P = (1, 0)$ by the ball $B(0; 1)$, and the function $f(x)$ by $2f(P)\,\chi(x)$, a multiple of the characteristic function of this ball, and by a localization argument we deduce that the behaviour of the means $S_R * f(x)$ in points of x of the form $(1 + \varepsilon, 0)$ is completely described by $2f(P)\, S_R * \chi(x)$.

The theorem is an immediate consequence of the following lemmas.

LEMMA 1. *Let*

$$\chi(\mathbf{x}) = \begin{cases} 1 & \textit{if } |\mathbf{x}| < 1, \\ \frac{1}{2} & \textit{if } |\mathbf{x}| = 1, \\ 0 & \textit{if } |\mathbf{x}| > 1. \end{cases}$$

Then if $|\mathbf{x}| > 1$ we have

$$S_R * \chi(\mathbf{x}) = \int_{R(|\mathbf{x}|-1)}^{+\infty} K(s)\, ds + E(\mathbf{x}, R),$$

and $E(\mathbf{x}, R)$ converges to zero as $R \to +\infty$, uniformly with respect to $\mathbf{x}$, $|\mathbf{x}| > 1$.

Proof. The Fourier transform of $\chi(x)$ is $\hat{\chi}(\xi) = |\xi|^{-N/2} J_{N/2}(2\pi|\xi|)$ and by the Fourier inversion formula for radial function we have

$$2\pi|x|^{-(N-2)/2} \int_0^{+\infty} J_{N/2}(2\pi t)\, J_{(N-2)/2}(2\pi|x|\, t)\, dt = \chi(x).$$

Hence, if $|x| > 1$,

$$S_R * \chi(x) = 2\pi R|x|^{-(N-2)/2} \int_0^{+\infty} [m(t)-1] \times J_{N/2}(2\pi Rt)\, J_{(N-2)/2}(2\pi R|x|\, t)\, dt.$$

The asymptotic expansion $J_\alpha(t) = \sqrt{2/\pi t}\cos(t - \alpha\pi/2 - \pi/4) + \mathcal{O}(t^{-3/2})$, yields

$$J_{N/2}(2\pi Rt)\, J_{(N-2)/2}(2\pi R|x|\, t) = \pi^{-2}|x|^{-1/2}\frac{\cos(2\pi Rt - \pi(N+1)/4)\cos(2\pi R|x|\, t - \pi(N-1)/4)}{Rt} + \mathcal{O}((Rt)^{-2}).$$

Splitting the interval of integration into $(O, R^{-1}) \cup (R^{-1}, +\infty)$, one realizes that the first interval gives a vanishing contribution, as $R \to +\infty$. Because of the assumptions on the function $m(t)$, the integral $R^{-1}\int_{R^{-1}}^{+\infty} |(m(t)-1)/t^2|\, dt$ vanishes also as $R \to +\infty$. Then

$$\begin{aligned}
S_R * \chi(x) &\approx \frac{2}{\pi}|x|^{-(N-1)/2}\int_0^{+\infty} [m(t)-1] \\
&\quad \times \cos(2\pi Rt - \pi(N+1)/4)\cos(2\pi R|x|\, t - \pi(N-1)/4)\frac{dt}{t} \\
&= |x|^{-(N-1)/2}\int_0^{+\infty} [1-m(t)]\frac{\sin(2\pi R(|x|-1)\, t)}{\pi t}\, dt \\
&\quad + |x|^{-(N-1)/2}\int_0^{+\infty} \frac{[m(t)-1]}{\pi t}\cos(2\pi R(|x|+1)\, t - \pi N/2)\, dt \\
&\approx |x|^{-(N-1)/2}\int_0^{+\infty} [1-m(t)]\frac{\sin(2\pi R(|x|-1)\, t)}{\pi t}\, dt \\
&= |x|^{-(N-1)/2}\left\{\frac{1}{2} - \frac{1}{2}\int_{-\infty}^{+\infty} m(|t|)\frac{\sin(2\pi R(|x|-1)\, t)}{\pi t}\, dt\right\}.
\end{aligned}$$

Recalling that $\sin(2\pi At)/\pi t$ is the Fourier transform of the characteristic function of the interval $[-A, +A]$ and that $m(|t|)$ is the Fourier transform of the function $K(s)$, we thus obtain,

$$\begin{aligned}
S_R * \chi(x) &\approx |x|^{-(N-1)/2}\left\{\tfrac{1}{2} - \tfrac{1}{2}\int_{-\infty}^{+\infty} K(s)\, \chi_{(-R(|x|-1),\, +R(|x|-1))}(s)\, ds\right\} \\
&= |x|^{-(N-1)/2}\int_{R(|x|-1)}^{+\infty} K(s)\, ds.
\end{aligned}$$

We finally observe that either $|x|$ is close to one, or the integral $\int_{R(|x|-1)}^{+\infty} K(s)\, ds$ is small. ∎

Let $S^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$ be the unit sphere in $\mathbb{R}^N$ equipped with normalized surface measure $d\sigma$. The spherical means $f(x; t)$ of a function $f(x)$ locally integrable on $\mathbb{R}^N$ are defined by

$$f(x; t) = \int_{S^{N-1}} f(x + t\sigma)\, d\sigma.$$

LEMMA 2. *Let $g(\mathbf{x}) = f(\mathbf{x}) - 2f(P)\,\chi(\mathbf{x})$. Then the spherical means $g(\mathbf{x}; t)$ at points $\mathbf{x} = (1 + \varepsilon, 0)$, ε suitably small, satisfy the estimates*

$$\left|\int_{S^{N-1}} g(\mathbf{x} + t\sigma)\, d\sigma\right| \leqslant c \begin{cases} \sqrt{t} & \text{if } 0 < t < 2\varepsilon, \\ t & \text{if } 2\varepsilon \leqslant t \leqslant 1, \\ 1 & \text{if } 1 < t. \end{cases}$$

The above constant can be taken independent of ε, i.e., of $\mathbf{x}$.

Proof. If we intersect a surface of equation $\{x_1 = 1 + |x_2|^2 P(x_2) + |x_2|^3 Q(x_2)\}$, $P(x_2)$ and $Q(x_2)$ suitable bounded functions in a neighbourhood of the origin, with a sphere of center $(1 + \varepsilon, 0)$ and radius t, with $0 < \varepsilon < t$ and ε and t suitably small,

$$\begin{cases} x_1 = 1 + |x_2|^2 P(x_2) + |x_2|^3 Q(x_2), \\ x_1 = 1 + \varepsilon + t\cos(2\pi\vartheta), \\ |x_2| = t\,|\sin(2\pi\vartheta)|, \end{cases}$$

we obtain as a solution $\cos(2\pi\vartheta) = -\varepsilon/t + \mathcal{O}(t)$, whence

$$\vartheta = \pm\frac{1}{2\pi}\operatorname{Arcos}(-\varepsilon/t) + \begin{cases} \mathcal{O}(\sqrt{t}) & \text{if } t \leqslant 2\varepsilon, \\ \mathcal{O}(t) & \text{if } 2\varepsilon < t. \end{cases}$$

Note that in the first approximation the angle ϑ does not depend on $P(x_2)$ and $Q(x_2)$, i.e., it is approximatively the same for every surface which is normal to the x_1 axis at the point $P = (0, 1)$.

If we apply this observation to the surface ∂D and to the sphere of center 0 and radius 1, we realize that the surface measure of the set of $\sigma \in S^{N-1}$ such that $x + t\sigma$ is in $D - B(0; 1)$ or in $B(0; 1) - D$ is small,

$$|\{\sigma \in S^{N-1} : x + t\sigma \in (D - B(0;1)) \cup (B(0;1) - D)\}| \leqslant c \begin{cases} \sqrt{t} & \text{if } t \leqslant 2\varepsilon \\ t & \text{if } 2\varepsilon < t. \end{cases}$$

On the other hand, in $D \cap B(0; 1)$ we have $|g(y)| \leqslant \|\nabla g\|_{L^\infty} |y-P|$, hence if $x + t\sigma$ is in $D \cap B(0; 1)$ we have that $|g(x+t\sigma)| \leqslant c\, t$.

Integrating these inequalities we obtain the thesis. ∎

LEMMA 3. *Let $g(\mathbf{x}) = f(\mathbf{x}) - 2f(P)\,\chi(\mathbf{x})$. Then if $\mathbf{x} = (1+\varepsilon, 0)$ and ε is suitably small we have that $\{S_R * g(\mathbf{x})\}$ converges to zero, as $R \to +\infty$. The convergence is uniform with respect to ε in a suitable right neighbourhood of* 0.

Proof. Using polar coordinates (and a small abuse of notation since S is radial) we can write the convolution $S_R * g(x)$ as

$$S_R * g(x) = \frac{2\pi^{N/2}}{\Gamma(N/2)} \int_0^{+\infty} R^N S(Rt)\, g(x; t)\, t^{N-1}\, dt.$$

Since, by our assumption, the kernel $S(x)$ belongs to $\mathbf{L}^1(\mathbb{R}^N)$, the lemma follows from the estimates on the spherical means $g(x; t)$ and standard arguments. ∎

For the proof of the previous theorem we assumed that the multiplier $(m(t)-1)/t$ is locally integrable and has Fourier transform vanishing at infinity. This is not disturbing, since for almost every reasonable function $m(t)$ we have these properties; e.g., take $m(t)$ Hölder continuous of any order and with compact support. We also assumed that the two kernels $K(x)$ and $S(\mathbf{x})$ are in $\mathbf{L}^1(\mathbb{R})$ and in $\mathbf{L}^1(\mathbb{R}^N)$, respectively. This is a more serious restriction, that could perhaps be weakened, but some assumptions on these two kernels have to be made. To explain this point we study now in more detail the Gibbs phenomenon for the Bochner–Riesz summability methods of multiple Fourier integrals.

2

The Bochner–Riesz means of order $\delta \geqslant 0$ of a test function $f(x)$ defined on $\mathbb{R}^N$ are defined via the Fourier transform by

$$(S_R^\delta * f)^\wedge(\xi) = (1 - R^{-2}|\xi|^2)_+^\delta\, \hat{f}(\xi).$$

These means can also be defined via the convolution with the radial kernels

$$S_R^\delta(x) = R^N \pi^{-\delta} \Gamma(\delta+1) |Rx|^{-\delta - N/2} J_{\delta + N/2}(2\pi |Rx|).$$

See [1, 7].

If $\delta > (N-1)/2$ (the "critical index" for the Bochner–Riesz summability), the kernels $\{S_R^\delta(x)\}$ belong to $\mathbf{L}^1(\mathbb{R}^N)$, and the hypothesis of the previous theorem is satisfied. In the next theorem we want to consider the Bochner–Riesz means of functions in the two-dimensional plane, below this critical index.

THEOREM 2. *Let D be a domain in $\mathbb{R}^2$, bounded by a smooth simple closed curve ∂D. Assume that the point $P = (1, 0)$ lies on ∂D, and that the vector $\mathbf{n} = (1, 0)$ is outward normal to ∂D in P. Let $F(\mathbf{x})$ be a smooth function on $\mathbb{R}^2$, and let*

$$f(\mathbf{x}) = \begin{cases} F(\mathbf{x}) & \text{on } D, \\ \frac{1}{2} F(\mathbf{x}) & \text{on } \partial D, \\ 0 & \text{on } \mathbb{R}^2 - \bar{D}. \end{cases}$$

Then if $\delta \geqslant 0$, if ε is a positive small number, and if $\mathbf{x} = (1 + \varepsilon, 0)$, we have that

$$S_R^\delta * f(\mathbf{x}) = 2f(P) \cdot 2^{\delta - 1/2} \pi^{-1/2} \Gamma(\delta + 1) \int_{2\pi R(|\mathbf{x}| - 1)}^{+\infty} \times s^{-\delta - 1/2} J_{\delta + 1/2}(s)\, ds + E(\mathbf{x}, R),$$

and $E(\mathbf{x}, R)$ converges to zero as $R \to +\infty$, uniformly with respect to $\mathbf{x}$ in a suitably small neighborhood of P.

The proof of this theorem is only a bit more complicated than the proof of Theorem 1, and it is a consequence of the following lemmas.

LEMMA 4. *Let*

$$\chi(\mathbf{x}) = \begin{cases} 1 & \text{if } |\mathbf{x}| < 1, \\ \frac{1}{2} & \text{if } |\mathbf{x}| = 1, \\ 0 & \text{if } |\mathbf{x}| > 1. \end{cases}$$

Then, if $|\mathbf{x}| > 1$ we have

$$S_R^\delta * \chi(\mathbf{x}) = 2^{\delta - 1/2} \pi^{-1/2} \Gamma(\delta + 1) \int_{2\pi R(|\mathbf{x}| - 1)}^{+\infty} s^{-\delta - 1/2} J_{\delta + 1/2}(s)\, ds + E(\mathbf{x}, R),$$

and $E(\mathbf{x}, R)$ converges to zero as $R \to +\infty$, uniformly with respect to $\mathbf{x}$, $|\mathbf{x}| > 1$.

Proof. Apply Lemma 1 to the function $m(t) = (1 - t^2)_+^\delta$. ∎

LEMMA 5. *Let* $g(\mathbf{x}) = f(\mathbf{x}) - 2f(P)\,\chi(\mathbf{x})$. *Then the spherical means* $g(\mathbf{x}; t)$ *at points* $\mathbf{x} = (1+\varepsilon, 0)$, ε *suitably small, satisfy the estimates*

$$\left|\int_{-\pi}^{\pi} g(1+\varepsilon+t\cos(2\pi\vartheta),\, t\sin(2\pi\vartheta))\, d\vartheta\right| \leqslant c \begin{cases} \sqrt{t} & \text{if } 0 < t < 2\varepsilon, \\ t & \text{if } 2\varepsilon \leqslant t \leqslant 1, \\ 1 & \text{if } 1 < t. \end{cases}$$

The above constant can be taken independent of ε, *i.e., of* $\mathbf{x}$.

Proof. Same as that for Lemma 2. ∎

LEMMA 6. *The spherical means* $t \to g(\mathbf{x}; t)$ *may not be continuous functions of* t, *but they have bounded variation on* $\mathbb{R}^+$. *The total variation of these spherical means can be bounded independently of* $\mathbf{x}$.

Proof. The proof that the total variation of the spherical means of the characteristic function $\chi(x)$ is finite is immediate.

We now consider the spherical means $t \to f(x; t)$. The variation of these means is controlled by $\|\nabla f\|_{\mathrm{L}^\infty}$ and by $\|f\|_{\mathrm{L}^\infty}$ and the variation of the domain of integration. Observe that if ∂D contains an arc of the circle with center x and radius t, the function $t \to f(x; t)$ may be discontinuous.

Suppose for simplicity that the intersection of the domain D with the circle of center (x_1, x_2) and radius t,

$$\begin{cases} x_1 + t\cos(2\pi\vartheta), \\ x_2 + t\sin(2\pi\vartheta), \end{cases}$$

consists of a single arc, i.e., $\psi(t) \leqslant \vartheta \leqslant \varphi(t)$. Then

$$\begin{aligned} &\left|\frac{d}{dt}\int_{\psi(t)}^{\varphi(t)} f(x_1 + t\cos(2\pi\vartheta), x_2 + t\sin(2\pi\vartheta))\, d\vartheta\right| \\ &\quad \leqslant \int_{\psi(t)}^{\varphi(t)} |\nabla f(x_1 + \cos(2\pi\vartheta), x_2 + t\sin(2\pi\vartheta))|\, d\vartheta \\ &\quad + |f(x_1 + t\cos(2\pi\psi(t)), x_2 + t\sin(2\pi\psi(t)))| \left|\frac{d}{dt}\psi(t)\right| \\ &\quad + |f(x_1 + t\cos(2\pi\varphi(t)), x_2 + t\sin(2\pi\varphi(t)))| \left|\frac{d}{dt}\varphi(t)\right|. \end{aligned}$$

The proof of the previous lemma shows that when t is suitably small, $\psi(t)$ and $\varphi(t)$ are piecewise monotone. In general $|d\psi|$ and $|d\varphi|$ are dominated by $ds/2\pi t$, where ds is the length element of ∂D. Hence the two functions $\psi(t)$ and $\varphi(t)$ are of bounded variation. ∎

LEMMA 7. *Let* $g(\mathbf{x}) = f(\mathbf{x}) - 2f(P)\,\chi(\mathbf{x})$. *Then if* $\mathbf{x} = (1+\varepsilon, 0)$ *and* ε *is suitably small we have that*

$$S_R^\delta * g(\mathbf{x}) = 2\pi^{1-\delta}\Gamma(\delta+1)\,R^{1-\delta}\int_0^{+\infty} t^{-\delta} J_{\delta+1}(2\pi Rt)\,g(\mathbf{x}; t)\,dt$$

converges to zero, as $R \to +\infty$. *The convergence is uniform with respect to* ε *in a suitable right neighbourhood of* 0.

Proof. Let η be a small positive constant. Using the estimate $|J_{\delta+1}(s)| \leqslant c\,s^{\delta+1}$ we have

$$\begin{aligned} R^{1-\delta} &\int_0^{R^{\eta-1}} t^{-\delta}\,|J_{\delta+1}(2\pi Rt)|\,|g(x; t)|\,dt \\ &\leqslant c\,R^2 \int_0^{R^{\eta-1}} |g(x; t)|\,t\,dt \\ &\leqslant c\,R^{2\eta} \sup_{0<t<R^{\eta-1}} |g(x; t)|. \end{aligned}$$

By the estimates on $|g(x; t)|$ in the previous lemmas the above quantity goes to zero as $R \to +\infty$.

We estimate the remaining part of the integral using the oscillatory nature of Bessel functions and the fact that the function $g(x; t)$ is of bounded variation in t. Define

$$A(t) = -\int_t^{+\infty} s^{-\delta} J_{\delta+1}(2\pi s)\,ds.$$

By the asymptotics of Bessel functions this (improper) integral is well defined, and

$$R^{1-\delta}\int_{R^{\eta-1}}^{+\infty} t^{-\delta} J_{\delta+1}(2\pi Rt)\,g(x; t)\,dt = \int_{R^{\eta-1}}^{+\infty} g(x; t)\,\frac{d}{dt}\,[A(Rt)]\,dt.$$

Since $A(t)$ converges to zero as $t \to +\infty$, and since, by the previous lemmas, the function $g(x; t)$ is of bounded variation in t, an integration by parts shows that the above integral vanishes as $R \to +\infty$. ∎

FINAL REMARKS

Remark 1. Let f be a bounded function with support in the N-dimensional torus $\mathbb{T}^N \cong \{\mathbf{x} : -\frac{1}{2} \leqslant x_i < \frac{1}{2}\}$. Then, besides the Fourier integrals

$$S_R * f(x) = \int_{\mathbb{R}^N} \hat{S}(R^{-1}t)\hat{f}(t)\exp(2\pi i x \cdot t)\,dt,$$

one can also consider the Fourier series

$$\tilde{S}_R * f(x) = \sum_{j \in \mathbb{Z}^N} \hat{S}(R^{-1}j)\hat{f}(j)\exp(2\pi i x \cdot j).$$

Under the integrability assumption for $S(x)$, the Poisson summation formula gives, for x in $\mathbb{T}^N$, the equiconvergence

$$|S_R * f(x) - \tilde{S}_R * f(x)| \to 0 \qquad \text{as} \quad R \to +\infty.$$

It is thus possible to reformulate Theorem 1 for Fourier series.

Remark 2. By Theorem 2 the overshooting in the Gibbs phenomenon for the Bochner–Riesz summability of order δ is described by the function

$$\Phi_\delta(t) = 2^{\delta+1/2}\pi^{-1/2}\Gamma(\delta+1)\int_t^{+\infty} s^{-\delta-1/2}J_{\delta+1/2}(s)\,ds.$$

Observe that when $\delta = 0$, $\Phi_\delta(t) = \frac{2}{\pi}\int_t^{+\infty}(\sin(s)/s)\,ds$, so that the Gibbs phenomenon for double Fourier integrals is quantitatively the same as the Gibbs phenomenon for Fourier series.

Here are some properties of the functions $\Phi_\delta(t)$.

(i) The maxima and minima of the function $\Phi_\delta(t)$ are attained at the zeroes of the Bessel function $J_{\delta+1/2}(t)$. As one may reasonably expect, the abosolute minimum is attained at the first positive zero, say $z(\delta+1/2)$, of $J_{\delta+1/2}(t)$; for a proof see [3].

(ii) The absolute minimum $\Phi_\delta(z(\delta+1/2))$ increases with δ. To see this we use the formula [8]

$$s^{-\delta-\nu-1/2}J_{\delta+\nu+1/2}(s) = \frac{2^{1-\nu}}{\Gamma(\nu)}\int_0^1 (su)^{-\delta-1/2}J_{\delta+1/2}(su)(1-u^2)^{\nu-1}u^{2\delta+2}\,du$$

to obtain

$$\begin{aligned}
\Phi_{\delta+\nu}(t) &= 2^{\delta+\nu+1/2}\pi^{-1/2}\Gamma(\delta+\nu+1)\int_t^{+\infty} s^{-\delta-\nu-1/2}J_{\delta+\nu+1/2}(s)\,ds \\
&= 2\frac{\Gamma(\delta+\nu+1)}{\Gamma(\delta+1)\Gamma(\nu)}\int_0^1 (1-u^2)^{\nu-1}u^{2\delta+1} \\
&\quad \times \left\{2^{\delta+1/2}\pi^{-1/2}\Gamma(\delta+1)\int_{tu}^{+\infty} s^{-\delta-1/2}J_{\delta+1/2}(s)\,ds\right\}du \\
&= \frac{\Gamma(\delta+\nu+1)}{\Gamma(\delta+1)\Gamma(\nu)}\int_0^1 (1-y)^{\nu-1}y^\delta\Phi_\delta(t\sqrt{y})\,dy.
\end{aligned}$$

Since,

$$\frac{\Gamma(\delta+\nu+1)}{\Gamma(\delta+1)\,\Gamma(\nu)}\int_0^1 (1-y)^{\nu-1}\,y^{\delta}\,dy=1,$$

$\Phi_{\delta+\nu}(t)$, $\nu>0$, is a convex average of the values $\Phi_\delta(t\sqrt{y})$, $0<y<1$. This implies that the minimum value of the function $\Phi_{\delta+\nu}(t)$ is greater than the minimum value of the function $\Phi_\delta(t)$.

(iii) The Gibbs phenomenon disappears as $\delta\to+\infty$. Indeed, it is known [8] that $z(\delta+1/2)>\delta+1/2$ and $|\mathbf{J}_{\delta+1/2}(s)|\leqslant 1$. This implies

$$\begin{aligned}|\Phi_\delta(z(\delta+1/2))| &\leqslant 2^{\delta+1/2}\pi^{-1/2}\Gamma(\delta+1)\int_{\delta+1/2}^{+\infty} s^{-\delta-1/2}\,ds\\ &\leqslant c(2/e)^{\delta}.\end{aligned}$$

Remark 3. Theorem 2 does not immediately extend to dimensions higher than two. The reason is essentially a lack of localization and convergence for the Bochner–Riesz summability of small order δ.

Indeed, suppose we have a closed surface in $\mathbb{R}^N$ and a point P on this surface. Also suppose that this surface contains a large piece of a sphere with center P and radius r. Then the spherical means $f(P,t)$ of a function which is different from zero inside the surface and equal to zero outside may not be continuous at $t=r$. Now consider the Bochner–Riesz means

$$S_R^{\delta}*f(x)=R^{N/2-\delta}\,\frac{2\pi^{N/2-\delta}\Gamma(\delta+1)}{\Gamma(N/2)}\int_0^{+\infty} f(x,t)\,J_{\delta+N/2}(2\pi Rt)\,t^{N/2-\delta-1}\,dt.$$

When $x\approx P$ and $t\approx r$ the integral gives a contribution which is of the order of $R^{-3/2}$, $R^{-1/2}$ because of the decay of the Bessel function $J_{\delta+N/2}(2\pi Rt)$ and R^{-1} because of the jump of $f(x,t)$. Thus it is not enough to cancel the factor $R^{N/2-\delta}$ in front of the integral if $N\geqslant 3$ and $\delta\leqslant(N-3)/2$.

In particular, when $N=3$ and $\delta=0$ for the partial sums of the Fourier integrals of the characteristic function of the unit ball in the origin we have

$$\begin{aligned}S_R^0*\chi_{\{|x|\leqslant 1\}}(0)&=R^3\int_{\{|y|\leqslant 1\}}|Ry|^{-3/2}J_{3/2}(2\pi|Ry|)\,dy\\ &=4R\int_0^1\left[\frac{\sin(2\pi Rt)}{2\pi Rt}-\cos(2\pi Rt)\right]dt\approx 1-\frac{2}{\pi}\sin(2\pi R).\end{aligned}$$

Note added in proof. L. De Michele and D. Roux have obtained a more general version of Theorem 1 for kernels which are not radial.

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